

The Semiring of Values of an Algebroid Curve

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Abstract

We introduce the semiring of values Γ with respect to the tropical operations associated to an algebroid curve. As a set, Γ determines and is determined by the well known semigroup of values S and we prove that Γ is always finitely generated in contrast to S . In particular, for a plane curve, we present a straightforward way to obtain Γ in terms of the semiring of each branch of the curve and the mutual intersection multiplicity of its branches. In the analytical case, this allows us to connect directly the results of Zariski and Waldi that characterize the topological type of the curve.

1 Introduction

Let \mathbb{K} be an algebraically closed field. We denote by $\mathbb{K}[[\underline{X}]]$ the ring $\mathbb{K}[[X_1, \dots, X_n]]$ of formal power series in the indeterminates X_1, \dots, X_n with coefficients in the field \mathbb{K} .

In this paper, an *algebroid curve* in the n -dimensional space \mathbb{K}^n ($n > 1$) is a proper radical ideal $Q = \bigcap_{i=1}^r P_i \subset \mathbb{K}[[\underline{X}]]$ such that $\mathcal{O}_i = \frac{\mathbb{K}[[\underline{X}]]}{P_i}$ has Krull dimension one for each isolated prime P_i with $i \in \{1, \dots, r\}$. Each P_i is called a *branch* of the curve Q . We will assume that the curve Q is *non-degenerate*, that is, $\dim_{\mathbb{K}} \frac{\mathcal{M}}{\mathcal{M}^2} = n$, where \mathcal{M} denotes the maximal ideal of the local, complete and reduced ring $\mathcal{O} = \frac{\mathbb{K}[[\underline{X}]]}{Q}$.

The integral closure $\overline{\mathcal{O}}_i$ of \mathcal{O}_i in its quotient field is a discrete valuation domain isomorphic to the ring $\mathbb{K}[[t_i]]$ and we have the inclusions (via isomorphism)

$$\mathcal{O} \subseteq \bigoplus_{i=1}^r \mathcal{O}_i \subseteq \overline{\mathcal{O}} = \bigoplus_{i=1}^r \overline{\mathcal{O}}_i = \bigoplus_{i=1}^r \mathbb{K}[[t_i]], \quad (1)$$

where $\overline{\mathcal{O}}$ denotes the integral closure of \mathcal{O} in its total ring of fractions.

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If $v_i : \overline{\mathcal{O}}_i \rightarrow \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ denotes the normalized discrete valuation of $\overline{\mathcal{O}}_i$, where $v_i(0) = \infty$, for all $i = 1, \dots, r$, then the set $S_i := \{v_i(g); g \in \mathcal{O}_i \setminus \{0\}\} \subseteq \mathbb{N}$ is classically called the *semigroup of values* of \mathcal{O}_i . Given a nonzero divisor $g \in \mathcal{O}$, we define $v(g) := (v_1(g), \dots, v_r(g)) \in \mathbb{N}^r$, where $v_i(g)$ means the value of the homomorphic image of $g \in \mathcal{O}$ in \mathcal{O}_i . In this way, we obtain the *semigroup of values* of \mathcal{O} :

$$S := \{v(g); g \text{ is a nonzero divisor in } \mathcal{O}\} \subseteq \bigoplus_{i=1}^r S_i \subseteq \mathbb{N}^r.$$

In [3] it is described a method to obtain S_i for a branch of a space curve and in [8] it is presented algorithms to compute S_i and sets of values for any \mathcal{O}_i -modules in $\overline{\mathcal{O}}_i$ as well.

Despite the fact semigroups of irreducible curves are finitely generated, the same is not true for curves with several branches. For instance, the semigroup of $Q = \langle XY \rangle = \langle X \rangle \cap \langle Y \rangle$ is $S = \{(0, 0)\} \cup \{(1, 1) + (\alpha_1, \alpha_2); (\alpha_1, \alpha_2) \in \mathbb{N}^2\}$ and it does not admit a finite set of generators as an additive semigroup.

For an analytic plane curve, i.e., $n = 2$, given by $Q = \langle f \rangle$, where $f = \prod_{i=1}^r f_i \in \mathcal{M} \subset \mathbb{C}\{X, Y\}$, Zariski (see [10] and [11]) shows that the topological type of $f^{-1}(0)$ is completely characterized by the semigroup S_i of each branch $\langle f_i \rangle$, with $1 \leq i \leq r$, and the mutual intersection multiplicity $I(f_j, f_k) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{X, Y\}}{\langle f_j, f_k \rangle}$, with $1 \leq j < k \leq r$. By the other hand, Waldi in [9] obtains the topological characterization of the germ $f^{-1}(0)$ by means of the semigroup S . In this way, a natural question is:

Question 1. *How to obtain S by means of S_i and $I(f_j, f_k)$, for $1 \leq i \leq r$ and $1 \leq j < k \leq r$?*

For two plane branches, Garcia and Bayer (see [7] and [1]) answer this question using the notion of *maximal points* of S . For a plane curve with several branches $Q = \cap_{i=1}^r \langle f_i \rangle$, Delgado in [5] determines S using the *relative maximal points* of S and the semigroups of $\cap_{i=1, i \neq j}^r \langle f_i \rangle$ for all $1 \leq j \leq r$.

Given an algebroid curve Q em \mathbb{K}^n , we propose to consider the set

$$\Gamma = \{v(g); g \in \mathcal{O}\} \supset S,$$

where $v(0) = \underline{\infty} := (\infty, \dots, \infty)$.

Obviously, $(\Gamma, +)$ is a semigroup setting $\gamma + \underline{\infty} = \underline{\infty}$ for all $\gamma \in \Gamma$. As a semigroup, Γ is the topological closure of S in the product topological space $\overline{\mathbb{N}}^r$, with $\overline{\mathbb{N}}$ provided of the one point compactification topology. This completion was considered for plane curves by Delgado (see Section 2 of [5]). In this paper, we will consider the set Γ equipped with the tropical operations

$$\alpha \oplus \beta = \min\{\alpha, \beta\} := (\min\{\alpha_1, \beta_1\}, \dots, \min\{\alpha_r, \beta_r\}) \quad \text{and} \quad \alpha \odot \beta = \alpha + \beta,$$

for all $\alpha = (\alpha_1, \dots, \alpha_r) \in \beta = (\beta_1, \dots, \beta_r) \in \Gamma$.

The main advantage of this approach is that (Γ, \oplus, \odot) is a *finitely generated semiring* and, for plane curves, its minimal set of generators allows us to connect S with S_i , for $1 \leq i \leq r$, and $I(f_j, f_k)$, for $1 \leq j < k \leq r$, that is, we obtain an answer to Question 1.

2 The Semiring of Values and Standard Bases

Throughout this paper, we denote the set of indices $\{1, \dots, r\}$ by I .

Let $Q = \bigcap_{i \in I} P_i \subset \mathbb{K}[[\underline{X}]]$ be an algebroid curve. As we remarked in the introduction, the integral closure $\overline{\mathcal{O}}_i$ of the domain $\mathcal{O}_i = \frac{\mathbb{K}[[\underline{X}]]}{P_i}$ is a discrete valuation ring isomorphic to $\mathbb{K}[[t_i]]$, where t_i is a uniformizing parameter of $\overline{\mathcal{O}}_i$. In what follows, we identify $X_j + P_i \in \mathcal{O}_i$ with its isomorphic image $x_j(t_i) \in \mathbb{K}[[t_i]]$ and \mathcal{O}_i with $\mathbb{K}[[x_1(t_i), \dots, x_n(t_i)]]$. We call $(x_1(t_i), \dots, x_n(t_i))$ a parameterization of P_i .

If v_i denotes the normalized valuation of $\overline{\mathcal{O}}_i$, we have the additive semigroup

$$\Gamma_i = v_i(\mathcal{O}_i) := \{v_i(g_i) = \text{ord}_{t_i}(g_i); g_i \in \mathcal{O}_i\} \subseteq \overline{\mathbb{N}},$$

setting $\gamma_i + \infty = \infty$ for all $\gamma_i \in \Gamma_i$ and $i \in I$.

Considering (1), we obtain the set of values of \mathcal{O} :

$$\Gamma = v(\mathcal{O}) := \{v(g) := (v_1(g), \dots, v_r(g)); g \in \mathcal{O}\} \subseteq \overline{\mathbb{N}}^r.$$

Notice that $S := \Gamma \cap \mathbb{N}^r$ and Γ determine each other.

Consider a non empty subset $J = \{j_1, \dots, j_s\}$ of I . Setting by Γ_J the set of values of $\mathcal{O}_J = \frac{\mathbb{K}[[\underline{X}]]}{\bigcap_{j \in J} P_j}$, we denote by Q^J the canonical image of the ideal $\bigcap_{i \in I \setminus J} P_i \subset \mathbb{K}[[\underline{X}]]$ in \mathcal{O}_J and by $v_J(Q^J)$ the Γ_J -monomodule $\{(v_{j_1}(q), \dots, v_{j_s}(q)); q \in Q^J\}$. If $J = \{i\}$, we put $Q^{\{i\}} = Q^i$ and $v_{\{i\}}(Q^{\{i\}}) = v_i(Q^i)$.

Since $\overline{\mathcal{O}}$ is an \mathcal{O} -module of finite type, the conductor $\mathcal{C} = (\mathcal{O} : \overline{\mathcal{O}})$ is an ideal of $\overline{\mathcal{O}}$ and of \mathcal{O} containing a nonzero divisor and $\mathcal{C} = (t_1^{\sigma_1}, \dots, t_r^{\sigma_r})\overline{\mathcal{O}}$. The element $\sigma = (\sigma_1, \dots, \sigma_r) \in \Gamma$ is called the *conductor* of Γ . As Γ_i has a conductor, there exists $\delta_i \in v_i(Q^i)$ such that $\delta_i + \overline{\mathbb{N}} \subseteq v_i(Q^i)$ and δ_i is the smallest element in $v_i(Q^i)$ with this property. D'Anna, in [4] (Proposition 1.3), proves that $\sigma_i = \delta_i$ for all $i \in I$.

In [8] it is presented an algorithm to compute the set of values for any finitely generated \mathcal{O}_i -module in $\mathbb{K}[[t_i]]$. In particular, we can obtain $v_i(Q^i)$ and compute σ_i for any $i \in I$.

Remark 2. For a plane curve $Q = \langle \prod_{i \in I} f_i \rangle = \bigcap_{i \in I} \langle f_i \rangle$, we have that $Q^i = \langle \prod_{\substack{j \in I \\ j \neq i}} f_j \rangle$ and

$$v_i(Q^i) = v_i \left(\prod_{\substack{j \in I \\ j \neq i}} f_j \right) + \Gamma_i = \sum_{\substack{j \in I \\ j \neq i}} I(f_j, f_i) + \Gamma_i.$$

In this way, we have the well know equality $\sigma_i = \sum_{\substack{j \in I \\ j \neq i}} I(f_j, f_i) + c_i$, where c_i is the conductor of Γ_i that can be computed in terms of the minimal set of generators of Γ_i .

Let $\alpha = (\alpha_1, \dots, \alpha_r)$ and $\beta = (\beta_1, \dots, \beta_r)$ be elements in Γ . We have the following properties:

- a) If $\alpha_i = 0$ for some $i \in I$, then $\alpha = \underline{0} := (0, \dots, 0)$.
- b) If $\alpha_k = \beta_k < \infty$ for some $k \in I$, then there exists $\gamma = (\gamma_1, \dots, \gamma_r) \in \Gamma$ such that $\gamma_i \geq \min\{\alpha_i, \beta_i\}$ for all $i \in I$ (the equality holds if $\alpha_i \neq \beta_i$) and $\gamma_k > \alpha_k = \beta_k$.
- c) $\min\{\alpha, \beta\} := (\min\{\alpha_1, \beta_1\}, \dots, \min\{\alpha_r, \beta_r\}) \in \Gamma$.

The last property allows us to consider Γ equipped with the tropical operations

$$\alpha \oplus \beta = \min\{\alpha, \beta\} \quad \text{and} \quad \alpha \odot \beta = \alpha + \beta.$$

It is immediate that (Γ, \oplus, \odot) is a semiring.

Definition 3. We call (Γ, \oplus, \odot) the semiring of values associated to the curve $Q = \bigcap_{i \in I} P_i$.

In the sequel we will show that Γ is a finitely generated semiring, that is, there exists a subset $\{\gamma_1, \dots, \gamma_m\} \subset \Gamma$ such that for any $\gamma \in \Gamma$ we can write

$$\gamma = (\gamma_1^{\alpha_{11}} \odot \dots \odot \gamma_m^{\alpha_{1m}}) \oplus \dots \oplus (\gamma_1^{\alpha_{s1}} \odot \dots \odot \gamma_m^{\alpha_{sm}}) = \min \left\{ \sum_{j=1}^m \alpha_{1j} \gamma_j, \dots, \sum_{j=1}^m \alpha_{sj} \gamma_j \right\}$$

with $\alpha_{ij} \in \mathbb{N}$, $1 \leq j \leq m$ and $1 \leq i \leq s$, for some $s \leq r$ which depends on γ .

Remark that for $r = 1$ we do not have novelty, so in what follows we always consider $r \geq 2$.

Let $G = \{g_1, \dots, g_m\}$ be a subset of $\mathcal{M} \subset \mathcal{O}$. A G -product is an element of the form

$$G^\alpha = \prod_{j=1}^m g_j^{\alpha_j},$$

with $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$.

From now on, given $\gamma \in \overline{\mathbb{N}}^r \setminus \{\underline{\infty}\}$ we put $I_\gamma = \{i \in I; \gamma_i \neq \infty\}$. If $g \in \mathcal{O} \setminus \{0\}$, then we denote $I_g = I_{v(g)}$.

Definition 4. Let g be a nonzero element in \mathcal{O} and consider $k \in I_g$. An element $h \in \mathcal{O}$ is a k -reduction of g modulo G if there exist $c \in \mathbb{K}$ and a G -product G^α such that

$$h = g - cG^\alpha,$$

with $v_i(h) \geq v_i(g)$ for all $i \in I$ and $v_k(h) > v_k(g)$. We say that h is a reduction of g modulo G if h is a k -reduction of g modulo G for some $k \in I_g$.

Remark 5. Notice that if $g \in \bigcap_{\substack{i \in I \\ i \neq k}} P_i \setminus P_k$ admits a reduction modulo G , then g admits only a k -reduction (because $I_g = \{k\}$) and there exists a G -product G^α such that $v(g) = v(G^\alpha)$.

Now we are able to introduce the notion of a Standard Basis for \mathcal{O} .

Definition 6. Let G be a nonempty finite subset of $\mathcal{M} \setminus \{0\}$. We say that G is a Standard Basis for \mathcal{O} if every nonzero element $g \in \mathcal{O}$ has a reduction modulo G .

The next proposition allows us to present another characterization of a Standard Basis for the local ring \mathcal{O} . For this purpose we need the following lemma whose proof is analogous to Lemma 1.8 of [6].

Lemma 7. Let J be a nonempty subset of I and let $(\alpha_1, \dots, \alpha_r) \in \overline{\mathbb{N}}^r$ with $\sigma_j \leq \alpha_j < \infty$ for all $j \in J$. We have that $(\alpha_1, \dots, \alpha_r) \in \Gamma$ if and only if there exists $(\beta_1, \dots, \beta_r) \in \Gamma$, where $\beta_i = \alpha_i$ for all $i \in I \setminus J$ and $\beta_j = \infty$ for all $j \in J$.

Proposition 8. Let G be a nonempty and finite subset of $\mathcal{M} \setminus \{0\}$. The following statements are equivalent:

- (a) Every nonzero element $g \in \mathcal{O}$ has a k -reduction modulo G for all $k \in I_g$.
- (b) Every nonzero element $g \in \mathcal{O}$ has a reduction modulo G .

Proof: It is sufficient to prove (b) \Rightarrow (a). Suppose, contrary to our claim, that there exist a nonzero element $g \in \mathcal{O}$ and $k \in I_g$ such that g admits a reduction modulo G , but it does not have a k -reduction. We may assume that g just has a j -reduction for all j in a nonempty subset J of $I_g \setminus \{k\}$.

Now consider an element $h \in \mathcal{O}$ obtained from g via a finite chain of reductions modulo G such that h does not have a j -reduction modulo G or $\sigma_j \leq v_j(h) < \infty$ for all $j \in J$, where $\sigma = (\sigma_1, \dots, \sigma_r)$ is the conductor of Γ . Notice that $h \neq 0$ and $v_i(h) = v_i(g)$ for all $i \in I \setminus J$. In this way h does not admit an i -reduction for any $i \in I_g \setminus J$, otherwise the same would be true for g .

Let us consider $L \subseteq J$ such that h does not admit an i -reduction for all $i \in I_g \setminus L \supseteq I_g \setminus J$, that is, $\sigma_l \leq v_l(h) < \infty$ for all $l \in L$. If $L = \emptyset$, then h does not have a reduction modulo G , which is a contradiction. By the other hand, if $L \neq \emptyset$, Lemma 7 implies that there exists $h' \in \mathcal{O} \setminus \{0\}$ such that $v_i(h') = v_i(h)$ for all $i \in I \setminus L$ and $v_l(h') = \infty$ for all $l \in L$. But in this way, h' does not admit a reduction modulo G and we obtain a contradiction again. ■

As an immediate consequence of the concept of reduction and the above proposition, we have the following characterization for a Standard Basis for \mathcal{O} .

Corollary 9. Let G be a nonempty and finite subset of $\mathcal{M} \setminus \{0\}$. The following statements are equivalent:

- (a) G is a Standard Basis for \mathcal{O} .
- (b) For every nonzero element $g \in \mathcal{O}$ and for some $k \in I_g$, there exists a G -product G^α (which depends on k) such that $v_i(g) \leq v_i(G^\alpha)$ for all $i \in I$ and $v_k(g) = v_k(G^\alpha)$.

(c) For every nonzero element $g \in \mathcal{O}$ and for all $k \in I_g$, there exists a G -product G^α (which depends on k) such that $v_i(g) \leq v_i(G^\alpha)$ for all $i \in I$ and $v_k(g) = v_k(G^\alpha)$.

In [8], the notion of Standard Basis was introduced for branches and, in that case, its existence is immediate. The next theorem guarantees the existence of a Standard Basis for the local ring of any algebroid curve with several branches.

Theorem 10. *The local ring \mathcal{O} admits a Standard Basis.*

Proof: Let H be a subset of \mathcal{O} satisfying $v(H) = v(\mathcal{M}) = \Gamma \setminus \{\underline{0}\}$ such that $v(h) \notin v(H \setminus \{h\})$ for all $h \in H$ and set $B_0 := \{h \in H; v_i(h) < \sigma_i \text{ if } i \in I_h\}$.

For all $i \in I$, consider $B'_i, B''_i \subset \mathcal{O}$ such that the homomorphic image of B'_i and B''_i in \mathcal{O}_i are Standard Bases for \mathcal{O}_i and Q^i respectively, which can be computed as described in [8]. As the homomorphic image of any finite subset A of \mathcal{O} such that $v_i(A) = v_i(B''_i)$ is a Standard Basis for Q^i , we can take B''_i as a subset of the homomorphic image of $\bigcap_{\substack{j \in I \\ j \neq i}} P_j$ in \mathcal{O} , that is, $v_j(h) = \infty$ for all $h \in B''_i$ and $j \in I \setminus \{i\}$. Setting $B_i = B'_i \cup B''_i$, we will show that the finite set $G = \bigcup_{i=0}^r B_i$ is a Standard Basis for \mathcal{O} .

Let g be a nonzero element in \mathcal{O} .

If $v_i(g) < \sigma_i$ for all $i \in I_g$, then there exists a G -product G^α (more specifically a B_0 -product) such that $v(g) = v(G^\alpha)$.

If $\sigma_k \leq v_k(g)$ for some $k \in I_g$, then $v_k(g) \in v_k(Q^k)$. As the homomorphic image of $B'_k, B''_k \subset \mathcal{O}$ are Standard Bases for \mathcal{O}_k and Q^k respectively, there exists a G -product G^α (indeed, a B'_k -product B^β and $h_k \in B''_k$ with $G^\alpha = B^\beta h_k$) such that $v_k(g) = v_k(G^\alpha)$ and $v_i(g) \leq v_i(G^\alpha) = \infty$ for all $i \in I \setminus \{k\}$.

By the above corollary, we conclude that G is a Standard Basis for \mathcal{O} . ■

The above proposition allows us to conclude that the semiring of values associated to an algebroid curve is finitely generated.

Theorem 11. *The semiring Γ is generated by $v(G)$, where G is a Standard Basis for \mathcal{O} .*

Proof: Let $G = \{g_1, \dots, g_m\}$ be a Standard Basis for \mathcal{O} with $v(g_j) = \gamma_j = (\gamma_{j1}, \dots, \gamma_{jr}) \in \overline{\mathbb{N}}^r$, for $1 \leq j \leq m$.

Initially notice that $\underline{0} = \sum_{j=1}^m 0 \cdot \gamma_j = \gamma_1^0 \odot \dots \odot \gamma_m^0$.

By Remark 5, for each $0 \neq h_k \in Q^k$, there exists a G -product G^{β_k} such that $v(h_k) = v(G^{\beta_k})$. In this way,

$$\underline{\infty} = v(h_1) + v(h_2) = v(G^{\beta_1}) + v(G^{\beta_2}) = v(G^\alpha) = \sum_{j=1}^m \alpha_j \cdot \gamma_j = \gamma_1^{\alpha_1} \odot \dots \odot \gamma_m^{\alpha_m},$$

where $\beta_1 + \beta_2 = \alpha = (\alpha_1, \dots, \alpha_m)$.

Now, given $\rho = (\rho_1, \dots, \rho_r) \in \Gamma \setminus \{\underline{0}, \underline{\infty}\}$, there exists $g \in \mathcal{M} \setminus \{0\}$ such that $\rho = v(g)$.

If $I_g = \{i_1, \dots, i_s\}$ then, by Corollary 9, there exist $\alpha_{kj} \in \mathbb{N}$ with $1 \leq k \leq s$ and $1 \leq j \leq m$ such that

$$\rho_{i_k} = v_{i_k}(g_1^{\alpha_{k1}} \cdot \dots \cdot g_m^{\alpha_{km}}) \text{ and } \rho_i \leq v_i(g_1^{\alpha_{k1}} \cdot \dots \cdot g_m^{\alpha_{km}}), \text{ for all } i \in I \setminus \{i_k\}.$$

In this way, for $i_k \in I_g$, we have

$$\rho_{i_k} = \min \left\{ \sum_{j=1}^m \alpha_{1j} \gamma_{ji_k}, \dots, \sum_{j=1}^m \alpha_{sj} \gamma_{ji_k} \right\}.$$

Therefore,

$$\rho = \min \left\{ \sum_{j=1}^m \alpha_{1j} \gamma_j, \dots, \sum_{j=1}^m \alpha_{sj} \gamma_j \right\} = (\gamma_1^{\alpha_{11}} \odot \dots \odot \gamma_m^{\alpha_{1m}}) \oplus \dots \oplus (\gamma_1^{\alpha_{s1}} \odot \dots \odot \gamma_m^{\alpha_{sm}}),$$

that is, the semiring Γ is finitely generated by $v(G)$. ■

Remark 12. Consider the \mathcal{M} -adic topology on \mathcal{O} and let $G = \{g_1, \dots, g_m\}$ be a Standard Basis for \mathcal{O} . Given $g \in \mathcal{O} \setminus \{0\}$, we have a chain (possibly infinite) of reductions modulo G

$$h_0 = g, \quad h_i = h_{i-1} - c_i G^{\alpha_i}, \quad i > 0,$$

where $c_i \in \mathbb{K}$ and G^{α_i} is a G -product.

In case of an infinite chain of reductions, we get a sequence $s_k = \sum_{i=1}^k c_i G^{\alpha_i}$, $k \geq 1$ in \mathcal{O} . As we have $v(G^{\alpha_i}) \neq v(G^{\alpha_j})$ for $i \neq j$, the set $\{c_i G^{\alpha_i}; i \geq 1\}$ is summable and the sequence s_k is convergent in \mathcal{O} .

Since G is a Standard Basis for \mathcal{O} , every element $g \in \mathcal{O} \setminus \{0\}$ admits a chain of reductions modulo G to 0, that is, $g = \lim_{k \rightarrow \infty} \sum_{i=1}^k c_i G^{\alpha_i}$ or, equivalently, $\mathcal{O} = \mathbb{K}[[g_1, \dots, g_m]]$.

3 Minimal Standard Bases and Irreducible Absolute Points

The semigroup S of an irreducible algebroid curve admits a minimal set of generators V in the sense that every system of generators of S contains V . In this way, it is natural analyze this property for the semiring of any algebroid curve with several branches that, in turn, is closely related with properties of a Standard Basis for \mathcal{O} .

It is obvious that for any Standard Basis G for \mathcal{O} and for every nonzero element $g \in \mathcal{M}$, the set $G \cup \{g\}$ is also a Standard Basis for \mathcal{O} . So it is convenient to introduce the following definition.

Definition 13. Let G be a Standard Basis for \mathcal{O} . We say that G is minimal if for every $g \in G$ there does not exist a reduction of g modulo $G \setminus \{g\}$.

In the next proposition we will prove that from a Standard Basis G we can always get a minimal Standard Basis discarding elements $g \in G$ that admit some reduction modulo $G \setminus \{g\}$. This will guarantee the existence of a minimal Standard Basis for \mathcal{O} .

Proposition 14. Let G be a Standard Basis for \mathcal{O} . If $g \in G$ admits some reduction modulo $H = G \setminus \{g\}$, then H is a Standard Basis for \mathcal{O} .

Proof: Suppose that g admits a k -reduction modulo H for some $k \in I_g$, that is, there exist $c_1 \in \mathbb{K}$ and an H -product H^{α_1} such that $h = g - c_1 H^{\alpha_1}$ satisfies $v_i(h) \geq v_i(g)$ for all $i \in I$ and $v_k(h) > v_k(g)$. If $v_i(g) = v_i(H^{\alpha_1})$ for all $i \in I$, by Corollary 9, we have that g admits an i -reduction modulo H for all $i \in I_g$. Therefore, H is a Standard Basis for \mathcal{O} .

By the other hand, if there exists $j \in I_g$ such that $v_j(g) < v_j(H^{\alpha_1})$, then $j \in I_h$ and h admits a j -reduction modulo G , since G is a Standard Basis for \mathcal{O} . Consequently, there exist $c_2 \in \mathbb{K}$ and a G -product G^β such that the element $h' = h - c_2 G^\beta = g - c_1 H^{\alpha_1} - c_2 G^\beta$ satisfies $v_i(h') \geq v_i(h) \geq v_i(g)$ for all $i \in I$ and $v_j(h') > v_j(h) = v_j(g)$.

In this way, we must have $G^\beta = g$ or G^β is an H -product H^{α_2} .

If $G^\beta = g$, then $c_2 = 1$, $h' = -c_1 H^{\alpha_1}$ e $v_k(h') = v_k(H^{\alpha_1}) < v_k(h)$, which is a contradiction. It follows that $G^\beta = H^{\alpha_2}$ and we obtain $v_i(g) \leq v_i(h) \leq v_i(H^{\alpha_2})$ for all $i \in I$ and $v_j(g) = v_j(H^{\alpha_2})$. By Corollary 9, we conclude that g admits a j -reduction modulo H . Hence, H is a Standard Basis for \mathcal{O} . ■

It is easy to see that the elements in a minimal Standard Basis have pairwise distinct values. Moreover, we have the following result.

Proposition 15. If G and H are Standard Bases for \mathcal{O} with H minimal, then $v(H) \subseteq v(G)$. In particular, all the minimal Standard Bases for \mathcal{O} have the same set of values.

Proof: Let $H = \{h_1, \dots, h_s\}$ and $G = \{g_1, \dots, g_m\}$ be Standard Bases for \mathcal{O} such that H is minimal.

We will show that $v(h_l) \in v(G)$ for all $h_l \in H$.

Without loss of generality, we can consider $l = 1$. As G is a Standard Basis for \mathcal{O} , given $k \in I_{h_1}$ there exists a G -product $g_1^{\alpha_1} \cdot \dots \cdot g_m^{\alpha_m}$ (with $\alpha_j = 0$ if $v_k(g_j) = \infty$) such that

$$v_i(h_1) \leq v_i(g_1^{\alpha_1} \cdot \dots \cdot g_m^{\alpha_m}) \text{ for all } i \in I \text{ and } v_k(h_1) = v_k(g_1^{\alpha_1} \cdot \dots \cdot g_m^{\alpha_m}).$$

By the other hand, for each $j \in \{1, \dots, m\}$, with $v_k(g_j) \neq \infty$, there exist an H -product $h_1^{\beta_{j1}} \cdot \dots \cdot h_s^{\beta_{js}}$ such that

$$v_i(g_j) \leq v_i(h_1^{\beta_{j1}} \cdot \dots \cdot h_s^{\beta_{js}}) \text{ for all } i \in I \text{ and } v_k(g_j) = v_k(h_1^{\beta_{j1}} \cdot \dots \cdot h_s^{\beta_{js}}).$$

But, in this way, we have

$$v_i(h_1) \leq v_i(h_1^{\sum_{j=1}^m \alpha_j \beta_{j1}} \dots h_s^{\sum_{j=1}^m \alpha_j \beta_{js}}) \text{ for all } i \in I \text{ and } v_k(h_1) = v_k(h_1^{\sum_{j=1}^m \alpha_j \beta_{j1}} \dots h_s^{\sum_{j=1}^m \alpha_j \beta_{js}}).$$

In particular, we must have $\sum_{j=1}^m \alpha_j \beta_{j1} \leq 1$.

If $\sum_{j=1}^m \alpha_j \beta_{j1} = 0$, then h_1 admits a k -reduction modulo $H \setminus \{h_1\}$, which contradicts the fact that H is a minimal Standard Basis for \mathcal{O} .

It follows that $\sum_{j=1}^m \alpha_j \beta_{j1} = 1$ and $\sum_{j=1}^m \alpha_j \beta_{j2} = \dots = \sum_{j=1}^m \alpha_j \beta_{js} = 0$. So, there exists $j_0 \in \{1, \dots, m\}$ such that $\alpha_{j_0} = \beta_{j_01} = 1, \alpha_{j_0} \beta_{j_02} = \dots = \alpha_{j_0} \beta_{j_0s} = 0$ and, consequently, $\beta_{j_0l} = 0$ for $l = 2, \dots, s$. Then we obtain $v_k(g_{j_0}) = v_k(h_1)$ and $\alpha_j = 0$ for all $j \neq j_0$. In addition, $v_i(h_1) \leq v_i(g_{j_0}) \leq v_i(h_1)$.

Therefore, $v(h_1) = v(g_{j_0}) \in v(G)$. ■

By the above proposition, if G is a minimal Standard Basis for \mathcal{O} , then $v(G)$ is the unique minimal system of generators for the semiring of values Γ .

In what follows we will continue to explore the relationship between a Standard Basis G for \mathcal{O} and the semiring of values Γ .

Definition 16. An element $\gamma \in \Gamma \setminus \{\underline{0}\}$ is called irreducible if

$$\gamma = \alpha + \beta; \alpha, \beta \in \Gamma \Rightarrow \alpha = \gamma \text{ or } \beta = \gamma.$$

Notice that the value of any element in a minimal Standard Basis is an irreducible element of the semiring. The algebraic counterpart of this property is true as well, that is, every element in a minimal Standard Basis G is irreducible in \mathcal{O} . Indeed, if $g = g_1 g_2 \in G$, where $g_1, g_2 \in \mathcal{M}$, then $\gamma = v(g) = v(g_1) + v(g_2) = \alpha + \beta$, with $\alpha \neq \gamma \neq \beta$.

Given $\gamma \in \Gamma$ and a proper subset J of I_γ , we set:

$$F_J(\gamma) = \{\alpha \in \Gamma; \alpha_i > \gamma_i \text{ for } i \in I_\gamma \setminus J \text{ and } \alpha_j = \gamma_j \text{ for } j \notin I_\gamma \setminus J\}.$$

Remark 17. If $v(g) = \gamma \neq \underline{\infty}$ and $F_J(\gamma) \neq \emptyset$ for some proper subset J of I_γ , then for every $j \in J$ there exists a j -reduction of g modulo a Standard Basis G . By the other hand, if $F_J(\gamma) = \emptyset$ for all $\emptyset \neq J \subset I_\gamma$, then the only possibility of reduction of g modulo G is $h = g - cG^\alpha$, with $v_i(h) > v_i(g)$ for all $i \in I_\gamma$, i.e., $v(g) = v(G^\alpha)$.

Definition 18. We say that $\gamma \in \Gamma$ is an absolute (maximal) point of Γ if $F_J(\gamma) = \emptyset$ for every proper subset J of I_γ .

Notice that if I_γ has only one element, then there does not exist a proper subset J of I_γ such that $F_J(\gamma) \neq \emptyset$. In this way, vacuously, γ is considered an absolute point of Γ .

For $r = 1$, the previous definition is equivalent to say that every element in Γ is an absolute point and the minimal system of generators of Γ is precisely its set of irreducible absolute points.

Now we will show that, similarly to the irreducible case, Γ is a semiring minimally generated by its irreducible absolute points.

Theorem 19. *Let G be a Standard Basis for \mathcal{O} such that its elements have pairwise distinct values. Then G is minimal if and only if $v(G)$ is the set of irreducible absolute points of Γ .*

Proof: Assume that $G = \{g_1, \dots, g_m\}$ is a minimal Standard Basis for \mathcal{O} and consider $g \in G$. In particular, $v(g) \neq \underline{0}$ is irreducible.

If $v(g)$ is not an absolute point of Γ , then there exist $h \in \mathcal{O}$ and a nontrivial subset J of I_g such that $v_i(g) < v_i(h)$ for all $i \in I_g \setminus J$ and $v_j(g) = v_j(h)$ for all $j \notin I_g \setminus J$ or, equivalently, for all $j \in J \cup (I \setminus I_g)$. As G is a Standard Basis, for each $k \in J$ there exists a G -product G^α (depending on k) such that $v_i(h) \leq v_i(G^\alpha)$ for all $i \in I$ and $v_k(h) = v_k(G^\alpha)$. Hence, $v_i(g) \leq v_i(G^\alpha)$ for every $i \in I$ with $v_k(g) = v_k(G^\alpha)$ and $v_j(g) < v_j(G^\alpha)$ for $j \in I_g \setminus J$. But, in this way, G^α is a $G \setminus \{g\}$ -product, that is, there exists a reduction of g modulo $G \setminus \{g\}$, an absurd because G is minimal. Therefore, $v(g)$ is an irreducible absolute point of Γ .

Now, let $\gamma = v(g)$ be an irreducible absolute point of Γ . By Remark 17, there exists a G -product such that $v(g) = v(G^\alpha)$. Furthermore, since the element γ is irreducible, we must have $G^\alpha = g_1^0 \cdot \dots \cdot g_j^1 \cdot \dots \cdot g_m^0$, for some $1 \leq j \leq m$. Therefore, $\gamma = v(g_j) \in v(G)$.

Conversely, assume that $v(G)$ is the set of all irreducible absolute points of Γ and let g be an element of G . Remark 17 implies that $v(g) = v(G^\alpha)$ for some G -product G^α . Now, as the elements in G have pairwise distinct values and $v(g)$ is irreducible we must have $G^\alpha = g$, that is, g does not have a reduction modulo $G \setminus \{g\}$. Hence, G is a minimal Standard Basis for \mathcal{O} . ■

As an immediate consequence we have the following result.

Corollary 20. *If G is a finite non empty subset of \mathcal{M} such that $v(G)$ is precisely the set of irreducible absolute points of Γ , then G is a minimal Standard Basis for \mathcal{O} .*

We will present in [2] an algorithm that allows us to obtain a Standard Basis for \mathcal{O} and consequently the minimal system of generators for the semiring of values Γ of any algebroid curve in \mathbb{K}^n . However, for plane curves we can give a direct and more precisely description.

Let $Q = \cap_{i=1}^r \langle f_i \rangle$ be a plane curve and let S be its semigroup of values. Given $J \subseteq I$, if π_J denotes the natural projection of $\overline{\mathbb{N}}^r$ to the set of indices J , then $\pi_{\{i\}}(S) = S_i$ and the number of absolute points of $\pi_{\{j,k\}}(S)$ is precisely the intersection multiplicity $I(f_j, f_k)$ (see [7]). In this way, S determines S_i , for $i \in I$, and $I(f_j, f_k)$, for $1 \leq j < k \leq r$.

By the other hand, as we remarked in the introduction, Delgado in [5] characterizes S in terms of the semigroup of the curves $Q^j = \langle \prod_{i \in I, i \neq j} f_i \rangle$ and a finite subset R (the set of relative

maximal points) of S (see the Generation Theorem in [5]). In order to obtain R , it is computed the set of irreducible absolute points of S , which corresponds to the set A of values of curves with maximal contact with some branch of Q and, using a symmetry property with respect to the conductor of S , he gets a set that contains R and this set allows to apply the Generation Theorem. We notice that A is precisely the set of irreducible absolute points of $S = \Gamma \cap \mathbb{N}^r$ and it can be obtained by S_i and $I(f_j, f_k)$, for $i \in I$ and $1 \leq j < k \leq r$.

The next proposition describes the irreducible absolute points of Γ with some coordinate equal to ∞ in terms of $I(f_j, f_k)$ and, as the semiring Γ and the semigroup S determine each other, provides an answer to Question 1.

Proposition 21. *The set of irreducible absolute points of Γ with some infinite coordinate for a plane curve $Q = \cap_{i \in I} \langle f_i \rangle$ is $\{v(f_i); i \in I\}$.*

Proof: If $\gamma = v(f_i)$ is not an absolute point of Γ for some $i \in I$, then there exists a nontrivial subset J of $I_{f_i} = I \setminus \{i\}$ such that $F_J(\gamma) \neq \emptyset$, i.e., there exists $\beta \in \Gamma$ such that $\beta_j = \gamma_j$ for all $j \in J$, $\beta_i = \gamma_i = \infty$ and $\beta_k > \gamma_k$ for all $k \notin J$ and $k \neq i$. As $\beta_i = \gamma_i = \infty$, we conclude that $\beta = v(hf_i)$ for some $h \in \mathcal{O}$. By the other hand, the equality $\beta_j = \gamma_j$, for all $j \in J$, implies $v(h) = \underline{0}$, but, in this way, $\beta = \gamma$, which is an absurd because $\beta_k > \gamma_k$ for all $k \notin J$ and $k \neq i$. Hence, $v(f_i)$ is an absolute point of Γ . It is immediate that $v(f_i)$ is irreducible, because f_i is irreducible.

Now, if $\gamma = v(g) \notin \mathbb{N}^r$ is an irreducible absolute point of Γ . Setting $K = I \setminus I_\gamma$ we conclude that $g = h \cdot \prod_{k \in K} f_k^{\alpha_k}$ and $\gamma = v(g) = v(h) + \sum_{k \in K} \alpha_k \cdot v(f_k)$. As γ is irreducible, we must have $v(h) = \underline{0}$, $K = \{k_0\}$ and $\alpha_{k_0} = 1$, that is, $\gamma = v(f_{k_0})$. ■

For the curve $Q = \langle XY \rangle$ the only irreducible absolute points of Γ are $\gamma_1 = v(y) = (1, \infty)$ e $\gamma_2 = v(x) = (\infty, 1)$, that is, $G = \{x, y\}$ is a minimal Standard Basis for \mathcal{O} and its semiring is $\Gamma = \{(0, 0)\} \cup \{(1, 1) + (\alpha_1, \alpha_2); (\alpha_1, \alpha_2) \in \overline{\mathbb{N}}^2\}$. Notice that any element $(\beta_1, \beta_2) \in \Gamma \setminus \{\underline{\infty}\}$ is obtained as

$$(\beta_1, \beta_2) = \min\{\beta_1(1, \infty), \beta_2(\infty, 1)\} = \gamma_1^{\beta_1} \oplus \gamma_2^{\beta_2}.$$

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